Utilization of the Karnaugh Map in Exploring Cause-effect Relations Modeled by Partially-defined Boolean Functions

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Authors’ contributions

This work was carried out in collaboration between both authors. Author AMAR designed the study, performed the analysis, solved the example, contributed to the literature search, and wrote the preliminary manuscript. Author RSB managed the literature search, and helped in solving the example. Both authors read and approved the final manuscript.

ABSTRACT

This paper utilizes a modern regular and modular eight-variable Karnaugh map in a systematic investigation of cause-effect relationships modeled by partially-defined Boolean functions (PDBF) (known also as incompletely specified switching functions). First, we present a Karnaugh-map test that can decide whether a certain variable must be included in a set of supporting variables of the function, and, otherwise, might enforce the exclusion of this variable from such a set. This exclusion is attained via certain don’t-care assignments that ensure the equivalence of the Boolean quotient w.r.t. the variable, and that w.r.t. its complement, i.e., the exact matching of the half map representing the internal region of the variable, and the remaining half map representing the external region of the variable, in which case any of these two half maps replaces the original full map as a representation of the function. Such a variable exclusion might be continued w.r.t. other variables till a minimal set of supporting variables is reached. The paper addresses a dominantly-unspecified PDBF to obtain all its minimal sets of supporting variables without resort to integer programming techniques. For each of the minimal sets obtained, standard map methods for...
extracting prime implicants allow the construction of all irredundant disjunctive forms (IDFs). According to this scheme of first identifying a minimal set of supporting variables, we avoid the task of drawing prime-implicant loops on the initial eight-variable map, and postpone this task till the map is dramatically reduced in size. The procedure outlined herein has important ramifications for the newly-established discipline of Qualitative Comparative Analysis (QCA). These ramifications are not expected to be welcomed by the QCA community, since they clearly indicate that the too-often strong results claimed by QCA adherents need to be checked and scrutinized.

Keywords: Partially-defined boolean function; set of supporting variables; Karnaugh map; prime implicant; irredundant disjunctive form; minimal sum; complete sum.

1. INTRODUCTION

Boolean or logical analysis of data has been a hot topic of research in the past few decades [1-7]. Of a particular interest herein is a seminal paper, in which Crama et al. [1] dealt with the problem of “identifying the small subsets of plausible causes of a given effect, among a large set of factors including all the potential causes, along with many other (irrelevant) factors.” They described all the combinations of possible causes which can be identified on the basis of a limited number of observations. They warned that “when only partial observations are available, no method can provide definite answers” though they hoped that even then the cause-effect relationship can be narrowed down sufficiently to provide substantial guidance. A suitable framework for handling the problem under consideration is that of partially-defined Boolean functions, also called incompletely-specified Boolean functions [1,8-22]. The computational problems arising in this framework (such as deriving minimal sets of supporting variables or finding all irredundant sums for each of these sets) are NP-hard, but in most applications of interest, the number of explanatory factors and observations are expected to be reasonably small. Crama et al. [1] mentioned similarities of their problem to similar formulations in artificial intelligence and machine learning [23-33]. Their problem is also of a nature quite similar to that addressed by Qualitative Comparative Analysis (QCA) [34-43], but it has subtle differences with the problem of digital-circuit design [41].

Crama et al. [1] addressed a problem of eight presumed variables. Out of 256 configurations in their problem, only 7 configurations were observable, and hence they dealt with a Boolean function that is partially defined. Employing integer linear programming, they found that their problem has eight minimal sets of supporting variables, and for each of these minimal sets, they obtained (usually) several irredundant disjunctive forms (IDFs). Any of these numerous IDFs (23 in total) could be the correct solution of the given problem. Though the work in [1] has been highly cited (315 citations in Google Scholar as of May 2021), we believe that it did not produce the impact it ought to. In fact, it has not been applied as the standard method for solving QCA problems except in very few recent papers [41-43]. There are possibly two main reasons for this state of affairs. One reason is that mainstream QCA typically seeks a more powerful, highly appealing and less ambiguous (albeit unjustified) result. It prefers the minimal sum among the set of all possible resulting IDFs, and hence the question of selecting a set of supporting variables is addressed by it, if ever, only as an afterthought. The second reason is that the work in [1] did not use the standard language and tools of Boolean analysis (as used by the digital-design community). In this paper, we hope to address the issue of this second reason by offering a complete exposition of the main result in [1] in standard language, aided by the pictorial insight offered by the Karnaugh map. We obtain all minimal sets of supporting variables for the problem in [1] without resort to integer programming techniques. For each of the minimal sets obtained, we employ standard map methods for extracting prime implicants to construct all irredundant disjunctive forms (IDFs). According to this scheme of first identifying a minimal set of supporting variables, we avoid the task of drawing prime-implicant loops on the initial eight-variable map, and postpone this task till the map is dramatically reduced in size.

The organization of the rest of this paper is as follows. Section 2 presents the problem of Crama et al. [1] used as a current example herein. Section 3 discusses the construction and use of the Karnaugh map. It also introduces the regular and modular version of the Karnaugh map to be used herein. This map version can be (theoretically) extended to an arbitrary large number of variables, and includes all maps of smaller sizes as special cases. Section 4 revisits
the problem in [1] from a map perspective, and recovers all the results obtained in [1]. Section 5 concludes the paper.

2. A RUNNING EXAMPLE

In Table 1, we present the running example to be used throughout this paper, which is taken from Carma, et al. [1]. Table 1 is a truth table with a single endogenous factor (outcome) $f$ and a set of exogenous factors $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ renamed herein as $(Y_3, Y_2, Y_1, Y_0, Z_3, Z_2, Z_1, Z_0)$. The renaming is made to facilitate the reconstruction of the truth table as a Karnaugh map. The set of eight factors is split into two distinct sets of variables of the Karnaugh map, respectively. Each row in Table 1 represents a unique configuration that is characterized by a binary number $(Y_3Y_2Y_1Y_0Z_3Z_2Z_1Z_0)_2$ or a hexadecimal number $(H_1H_0)_{16}$, where

$$H_1 = 2^3Y_3 + 2^2Y_2 + 2^1Y_1 + 2^0Y_0 = 8Y_3 + 4Y_2 + 2Y_1 + Y_0, \quad (1)$$

$$H_0 = 2^3Z_3 + 2^2Z_2 + 2^1Z_1 + 2^0Z_0 = 8Z_3 + 4Z_2 + 2Z_1 + Z_0. \quad (2)$$

Note that the hexadecimal number system uses 16 symbols, the first 10 of which (0,1,2,...,9) are borrowed (with the same meaning) from the conventional decimal system, while the remaining 6 symbols are the beginning uppercase letters of the alphabet A, B, C, D, E, and F used to designate the values 10, 11, 12, 13, 14, and 15, respectively, in decimal, which correspond to 1010, 1011, 1100, 1101, 1110 and 1111, respectively in binary (Table 2).

Designation of a configuration in Table 1 by a hexadecimal number $H_1H_0$ considerably facilitates the conversion of the truth table in Table 1 to the Karnaugh map in Fig. 1. The configuration $H_1H_0$ is simply located at the horizontal coordinate $H_0$ and the vertical coordinate $H_1$. Note that the map input is given in terms of the shown designations of the $Y$ and $Z$ variables, or equivalently, by the corresponding values of $H_1$ and $H_0$ which are deduced from equations (1) and (2). A few remarks about this running example are in order.

- Through this example is suggested for a context other than those typically employed in QCA applications, it nevertheless represents a truth table for a partially-defined (incompletely-specified) Boolean function, i.e., a function with logical remainders (in QCA jargon).
- The truth table is notorious for its extremely few specifications. Out of $2^8 = 256$ possible configurations, data is available for only 7 configurations. The fact is vividly illustrated by Figs. 2 or 3, in each of which the Karnaugh map has 256 cells, with only 7 of them of assigned or asserted entries of 0 or 1. The remaining 249 cells are left blank, indicating that they are logical remainders (in QCA terminology) or don’t-cares (in the digital-design terminology).
- No consistency cutoff is assigned to the values asserted for $f$ in Table 1. We assume a perfect consistency of 1.00 for each of the 7 configurations specified in Table 1.

<table>
<thead>
<tr>
<th>$Y_3$</th>
<th>$Y_2$</th>
<th>$Y_1$</th>
<th>$Y_0$</th>
<th>$Z_3$</th>
<th>$Z_2$</th>
<th>$Z_1$</th>
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<td>0</td>
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<td>0</td>
<td>6</td>
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<td>1</td>
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<td>0</td>
<td>2</td>
<td>A</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Truth table of the running example, adapted from Crama et al. [1]. The truth table expresses a single endogenous factor (outcome) in terms of eight exogenous factors (inputs) expressed as dichotomous (binary) variables that can be compressed into two hexadecimal variables $H_1H_0$. Only 7 out of 256 lines or configurations are specified.
Table 2. Hexadecimal ‘digits’ in terms of decimal digits and binary bits

<table>
<thead>
<tr>
<th>Hexadecimal ‘Digits’</th>
<th>Decimal Digits</th>
<th>Binary Bits</th>
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</thead>
<tbody>
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<td>1101</td>
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<td>E</td>
<td>14</td>
<td>1110</td>
</tr>
<tr>
<td>F</td>
<td>15</td>
<td>1111</td>
</tr>
</tbody>
</table>

Fig. 1. A regular and modular eight-variable Karnaugh map

3. CONSTRUCTION AND USE OF THE KARNAUGH MAP

Construction of the Karnaugh map is based on repeated application of the Boole-Shannon Expansion [44-46], which takes the following form when implemented w.r.t. a single variable $X_k$:

$$f(X) = (\overline{X}_k \land f(X|0_k)) \lor (X_k \land f(X|1_k))$$

(3)

This Boole-Shannon Expansion expresses the Boolean function $f(X)$ in terms of its two subfunctions $f(X|0_k)$ and $f(X|1_k)$. These subfunctions are equal to the Boolean quotients $f(X)/\overline{X}_k$ and $f(X)/X_k$, and hence are obtained by restricting $X_k$ in the expression $f(X)$ to 0 and 1, respectively. If $f(X)$ is an expression of $n$ variables, the two subfunctions $f(X|0_k)$ and $f(X|1_k)$ are functions of at most $(n-1)$ variables. If the Boole-Shannon expansion is applied in sequence to the $n$ variables of $f(X)$,
the expansion tree is a complete binary tree (usually called a Binary Decision Diagram) of $2^n$ leaves. These leaves are functions of no variables, or constants, and equal the entries of a corresponding conventional Karnaugh map of $f(X)$. Sibling nodes (nodes at the same level) of this expansion tree constitute the entries of a variable-entered (or a map-entered) Karnaugh map of $f(X)$ [17,45]. Various types of map folding allow the replacement of an $n$-variable map by two $(n-1)$-variable maps [34,35,38,18,47-51]. Conversely, the $n$-variable map might be viewed as a map-entered map [52-55] with a new map variable, say $Z_n$, and two major cells, each of which having the size of an $(n-1)$-variable map. Such a view might be repeated recursively, so as to construct a map of any desirable size.

If a Karnaugh map is used to represent a Boolean function $f(X)$, then the map can be split into two halves (with respect to the borders of the variable $X_k$) representing the internal and external domains of this variable. These half maps depict, respectively, the two subfunctions or Boolean quotients $f(X)/\bar{X}_k$ and $f(X)/X_k$, which are functions of the $(n-1)$ variables of $X$ other than $X_k$. We say that $f(X)$ is vacuous in (independent of) $X_k$ if the following relation is identically satisfied [34,49,51,56].

$$f(X)/\bar{X}_k = f(X)/X_k.$$  \hspace{1cm} (4)

According to (3), if (4) is identically satisfied then $f(X)$ becomes equal to each of $f(X)/\bar{X}_k$ and $f(X)/X_k$, and hence it becomes a function of the $(n-1)$ variables of $X$ other than $X_k$. This means that $X_k$ is now guaranteed to be not a supporting variable of $f(X)$. The variable $X_k$ must be included in a set of supporting variables of $f(X)$ if at least a cell in the half map $f(X)/\bar{X}_k$ is found to differ in its entry from its image cell in the half map $f(X)/X_k$ (the cell obtained through reflection of the original cell with respect to the nearest border of $X_k$). Note that only the 0 and 1 entries are viewed as different, opposing or contradictory, while a don’t-care entry (d) can be made to match either 0 or 1. The variable $X_k$ might be excluded from a set of supporting variables of $f(X)$ if (a) no case of the aforementioned pair of contradictory mirror-image cells can be found, and (b) appropriate conditions are placed on don’t-care cells in either the $f(X)/\bar{X}_k$ or $f(X)/X_k$ half maps that are images of 0-entered or 1-entered cells in the other half so as to ensure that (4) is satisfied.

In this paper, we will use a regular and modular form of the Karnaugh map that appeared earlier in [41-43,57-69], and is such that

a) The rectangular shape of the map cell is retained.

b) The internal domain of the $(n+1)$st variable is introduced to be centered around the boundary lines of the $(n-1)$st variable.

We note that the process outlined in (b) above can be, in theory, indefinitely continued. Hence, there is no theoretical upper bound on the size of the Karnaugh map constructed this way. However, as the number of variables increases, the size of the map increases exponentially, and its utility diminishes very quickly due to prohibitively increasing difficulty. As a demonstration of the usefulness of the aforementioned version of the Karnaugh map, we present its case of eight variables herein. This map (shown in Fig. 1) is exactly what we need for our current example.

4. KARNAUGH MAP SOLUTION OF THE RUNNING EXAMPLE

Fig. 1 presents the regular and modular eight-variable 256-cell Karnaugh map used herein. Fig. 2 indicates locations of specified configurations of the running example given in Sec.2. These locations are taken from Table 1, and are depicted here as two hexadecimal numbers, which are colored green for high assertion (1) and red for low assertion (0). As usual (in digital-design circles), the unspecified cells or configurations are left blank, and are implicitly understood to be entered with don’t-cares (called logical remainders in QCA jargon). For convenience, the numerical values of the hexadecimal numbers are used to mark the vertical and horizontal axes of the map. Fig. 3 replaces the input variables $(Y_3, Y_2, Y_1, Y_0, Z_3, Z_2, Z_1, Z_0)$ in Fig. 1 by $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ to restore the original names or uniform symbols in Crama et al. [1], and identify the cells of high assertion (1) and low assertion (0) by their entries rather than their locations.

At this stage, we discover that our present example is much more difficult and versatile than the ones we encountered earlier [41-43]. The present example has fewer specified configurations, and none of its variables is an essential supporting variable, i.e., its outcome
function can be made independent of any single variable. For example, Fig. 4 manages to make \( f \) independent of the variable \( X_2 \) by imposing appropriate conditions for \( f \) in Fig. 3, so that cells that are equidistant from the borders of \( X_2 \) (highlighted in red) are ensured to have equal entries. Fig. 5 pushes this issue further through discovery that the outcome \( f \) can be made independent not only of \( X_2 \) but also of each of \( X_4, X_6 \) and \( X_8 \). This causes the outcome \( f \) to become dependent on members of the set of the four remaining variables \( \{X_1, X_3, X_5, X_7\} \), and hence proves that this set is a minimal set of supporting variables. In Fig. 6, the Karnaugh map of Fig. 5 is reduced to one of these four variables \( \{X_1, X_3, X_5, X_7\} \). This map inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has six IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant \( X_3 \overline{X}_7 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( \overline{X}_1 \overline{X}_3 \overline{X}_5 X_7 \) (colored blue) might be covered by any of the three four-cell loops \( \overline{X}_1 \overline{X}_3 \overline{X}_5 X_7 \), or \( \overline{X}_3 \overline{X}_5 X_7 \), while the 1 in the cell \( X_1 \overline{X}_3 \overline{X}_5 X_7 \) (colored pink) might be covered by any of the two two-cell loops \( X_1 \overline{X}_5 \overline{X}_7 \) and \( X_1 \overline{X}_5 X_7 \).

Fig. 7 takes another turn to prove that the alternative set of four variables \( \{X_1, X_4, X_5, X_7\} \) is yet another minimal set of supporting variables. This is achieved through discovery that the outcome \( f \) can be made independent of each of \( X_2, X_3, X_4 \) and \( X_8 \), and subsequent observation that this outcome becomes dependent on members of the afore-mentioned set. Fig. 8 reduces the Karnaugh map of Fig. 7 to one of the four variables \( \{X_1, X_4, X_5, X_7\} \). This map again inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has another set of six IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant \( \overline{X}_4 \overline{X}_7 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( \overline{X}_1 X_4 \overline{X}_5 X_7 \) (colored blue) might be covered by any of the three four-cell loops \( \overline{X}_1 \overline{X}_3 \overline{X}_5 X_7 \), or \( \overline{X}_3 \overline{X}_5 X_7 \), while the 1 in the cell \( X_1 \overline{X}_4 \overline{X}_5 X_7 \) (colored pink) might be covered by any of the two two-cell loops \( X_1 X_5 \overline{X}_7 \) or \( X_1 \overline{X}_4 X_5 \).

Fig. 9 now proves that the set of three variables \( \{X_1, X_2, X_3\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_3, X_4, X_6, X_7 \) and \( X_8 \) but then it becomes solely dependent on (and minimally supported by) members of that set. The result in this figure is much more elegant than the ones in Figs. 5 and 7 thanks to the perfect visual adjacency of the reduced cells and to the decreased cardinality of the supporting set. Fig. 10 reduces the Karnaugh map of Fig. 9 to one of the three variables \( \{X_1, X_2, X_3\} \). This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map (with two of the 0’s combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant \( X_2 \overline{X}_3 \) (highlighted in yellow) that covers the starred 1, and also covers the 1 in the cell above, thereby turning it into a don’t care (since, for further processing, this cell can (but does not have to) be covered). The 1 in the cell \( \overline{X}_1 \overline{X}_3 X_2 \overline{X}_5 \) (colored blue) might be covered by any of the two two-cell loops \( \overline{X}_1 \overline{X}_2 \) or \( \overline{X}_1 \overline{X}_5 \).

With three minimal sets of supporting variables so far discovered, Fig. 11 now adds a fourth one. This figure proves that the set of three variables \( \{X_2, X_4, X_6\} \) is such a fourth minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_3, X_5, X_7 \) and \( X_2 \) but then it becomes minimally supported by members of that set. Fig. 12 reduces the Karnaugh map of Fig. 9 to one of the three variables \( \{X_2, X_4, X_6\} \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with two of the 0’s combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant \( X_2 \overline{X}_4 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( X_2 \overline{X}_4 X_6 \) (colored blue) might be covered by any of the two two-cell loops \( X_2 \overline{X}_6 \) or \( \overline{X}_6 \).

Now, Fig. 13 acts as a proof that the set of three variables \( \{X_2, X_3, X_7\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_3, X_4, X_6 \), which makes it minimally supported by members of that set. Figure 14 reduces the Karnaugh map of Fig. 13 to one of the three variables \( \{X_2, X_3, X_7\} \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with two of the 0’s combined). With this minimal set of
supporting variables, the function has two more IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \( X_2 X_5 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( X_2 X_5 X_7 \) (colored blue) might be covered by any of the two two-cell loops \( X_2 X_7 \) or \( X_5 X_7 \).

Similarly to the few previous figures, Fig. 15 adds a proof that the set of three variables \( \{X_4, X_5, X_6\} \) is also a minimal set of supporting variables through discovery that the outcome \( f \) can be made independent of each of \( X_3, X_4, X_5, X_7 \) and \( X_9 \), followed by asserting that this outcome then becomes minimally supported by members of that set. Fig. 16 reduces the Karnaugh map of Fig. 15 to one of the three variables \( \{X_4, X_5, X_6\} \). This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant \( X_5 X_6 \) (highlighted in yellow) that covers the single starred 1, and another essential prime implicant \( X_1 X_2 \) that covers the double-starred one. The function has a single don’t-care cell, which is assigned 0 for the solution above. Under this assignment, the complement of the function also has a single IDF. The situation in Figs. 15 and 16 is the most appealing so far according to certain arguments [22,41-43,62], but its appealing or desirable nature is no excuse to let it dominate or rule out other possibilities.

Now, Fig. 17 proves that the set of two variables \( \{X_5, X_6\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_2, X_5, X_4, X_6 \) and \( X_7 \) but then it becomes minimally supported by members of that set. Fig. 18 reduces the Karnaugh map of Fig. 17 to one of the two variables \( \{X_5, X_6\} \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with three of the 0’s combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant \( X_5 X_6 \) (highlighted in yellow) that covers the starred 1, and another essential prime implicant \( X_5 X_6 \) (highlighted in green) that covers the double-starred 1. The situation in Figs. 19 and 20 resembles that in Figs. 17 and 18.

Now, we have completely recovered all the minimal sets of supporting sets obtained via integer linear programming, together with their associated sets of IDFs [1, Table 4]. Admittedly, we have been guided by our pre-knowledge of Table 4 in [1]. However, our heuristic procedure would work even without such pre-knowledge. Fig. 21 is an attempt to prove that the set of two variables \( \{X_1, X_2\} \) is a minimal set of supporting variables, which is doomed to failure. The figure shows that \( f \) can (for example) be made independent of each of \( X_3, X_4, X_5, X_7 \) and \( X_6 \) but then it becomes impossible to render it independent of \( X_5 \) because the two half maps across the \( X_5 \) borders (highlighted in red) cannot be matched. On the other hand, Fig. 22 attempts to prove that the set of four variables \( \{X_1, X_2, X_5, X_6\} \) is a minimal set of supporting variables. The function \( f \) can be made independent of each of \( X_3, X_4, X_6, X_7 \) and \( X_6 \) as shown but the reduced map of \( f \) in terms of \( X_1, X_2, X_5 \) and \( X_6 \) does not exhibit minimality of the set of supporting variables. In fact, Fig. 23 illustrates that the set of four supporting variables \( \{X_1, X_2, X_5, X_6\} \) is not a minimal one, as it can drop either \( X_5 \) or \( X_6 \), thereby being reduced to either the set \( \{X_1, X_2, X_5\} \) (same as the one in Fig. 10) or the set \( \{X_1, X_2, X_6\} \) (same as the one in Fig. 16). These two latter sets are minimal as each of them cannot drop further any variable.
Fig. 2. Locations of specified configurations of the running example, depicted as two hexadecimal numbers, and colored green for high assertion (1) and red for low assertion (0). For convenience, the numerical values of the hexadecimal numbers are shown.

Fig. 3. Replacing the input variables \((Y_3, Y_2, Y_1, Y_0, Z_3, Z_2, Z_1, Z_0)\) in Fig. 1 by \((X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)\) to restore the original names in Crama et al. [1], and identifying the cells of high assertion (1) and low assertion (0).
4. Imposing conditions for $f$ in Fig. 3 to make it independent of $X_2$. Cells that are equidistant from the borders of $X_2$ (highlighted in red) are ensured to have equal entries.

$$
\begin{array}{c|c|c|c|c|c|c|c|c|c}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

$f$

Fig. 4. Imposing conditions for $f$ in Fig. 3 to make it independent of $X_2$. Cells that are equidistant from the borders of $X_2$ (highlighted in red) are ensured to have equal entries.

5. Proof that the set of four variables $\{X_1, X_2, X_5, X_7\}$ is a minimal set of supporting variables through discovery that $f$ can be made independent of each of $X_2, X_4, X_6$ and $X_8$ but then it becomes dependent on members of that set.

$f$

Fig. 5. Proof that the set of four variables $\{X_1, X_2, X_5, X_7\}$ is a minimal set of supporting variables through discovery that $f$ can be made independent of each of $X_2, X_4, X_6$ and $X_8$ but then it becomes dependent on members of that set.
Fig. 6. The Karnaugh map of Fig. 5 reduced to one of the four variables \(\{X_1, X_3, X_5, X_7\}\). This map inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has six IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \(X_3\bar{X}_7\) (highlighted in yellow) that covers the starred 1. The 1 in the cell \(X_1\bar{X}_3\bar{X}_5\bar{X}_7\) (colored green) might be covered by any of the three four-cell loops \(X_1\bar{X}_3\bar{X}_5\bar{X}_7\), \(\bar{X}_1\bar{X}_3\bar{X}_5\bar{X}_7\) or \(X_3\bar{X}_5\bar{X}_7\), while the 1 in the cell \(\bar{X}_3X_5\bar{X}_7\) (colored pink) might be covered by any of the two two-cell loops \(X_1X_5\bar{X}_7\) and \(X_1\).

Fig. 7. Proof that the set of four variables \(\{X_1, X_3, X_5, X_7\}\) is a minimal set of supporting variables through discovery that \(f\) can be made independent of each of \(X_2, X_3, X_6\) and \(X_8\) but then it becomes dependent on members of that set.
Fig. 8. The Karnaugh map of Fig. 7 reduced to one of the four variables \( \{X_1, X_4, X_5, X_7\} \). This map again inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has another set of six IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \( X_4 X_7 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( X_1 X_3 X_4 X_5 X_7 \) (colored blue) might be covered by any of the three four-cell loops \( X_1 X_3 X_4 X_5 \), \( X_4 X_7 \), or \( X_5 X_7 \), while the 1 in the cell \( X_1 X_4 X_5 X_7 \) (colored pink) might be covered by any of the two two-cell loops \( X_1 X_4 \) and \( X_1 X_5 X_7 \).

\[
\begin{array}{cccc}
X_4 & 1 & 0^* \\
X_3 & 1^* & 0^* \\
X_2 & 0 & 1 \\
X_1 & & & \\
X_5 & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
X_7 & X_7 & & \\
X_7 & & & \\
X_7 & & & \\
X_7 & & & \\
X_7 & & & \\
\end{array}
\]

\[
f
\]

Fig. 9. Proof that the set of three variables \( \{X_1, X_2, X_3\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_3, X_4, X_5, X_7 \) and \( X_8 \) but then it becomes solely dependent on (and minimally supported by) members of that set. The result in this figure is much more elegant than the ones in Figs. 5 and 7 thanks to the perfect visual adjacency of the reduced cells and to the decreased cardinality of the supporting set.

\[
f
\]
Fig. 10. The Karnaugh map of Fig. 9 reduced to one of the three variables \( \{X_1, X_2, X_3\} \). This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map (with two of the 0’s combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \( X_2X_3 \) (highlighted in yellow) that covers the starred 1, and also covers the 1 in the cell above, thereby turning it into a don’t care. The 1 in the cell \( \bar{X}_1X_2 \bar{X}_3 \) (colored blue) might be covered by any of the two two-cell loops \( \bar{X}_1X_2 \) or \( \bar{X}_1X_3 \).

Fig. 11. Proof that the set of three variables \( \{X_2, X_6, X_8\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_3, X_4, X_5 \) and \( X_7 \) but then it becomes minimally supported by members of that set.
Fig. 12. The Karnaugh map of Fig. 9 reduced to one of the three variables \( \{X_2, X_6, X_8\} \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with two of the 0’s combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \( X_2 X_8 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( X_2 X_6 X_8 \) (colored blue) might be covered by any of the two two-cell loops \( X_2 X_6 \) or \( X_6 X_8 \).

Fig. 13. Proof that the set of three variables \( \{X_2, X_5, X_7\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_3, X_4, X_6 \), and \( X_8 \) but then it becomes minimally supported by members of that set.
Fig. 14. The Karnaugh map of Fig. 13 reduced to one of the three variables \( \{X_2, X_5, X_7\} \). This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two more IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant \( X_2 \) (highlighted in yellow) that covers the starred 1. The 1 in the cell \( X_2 X_5 X_7 \) (colored blue) might be covered by any of the two two-cell loops \( X_2 X_7 \) or \( X_5 X_7 \).

Fig. 15. Proof that the set of three variables \( \{X_1, X_2, X_6\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_3, X_4, X_5, X_7 \) and \( X_8 \) but then it becomes minimally supported by members of that set.
Fig. 16. The Karnaugh map of Fig. 15 reduced to one of the three variables \(\{X_1, X_2, X_6\}\). This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant \(X_2 \overline{X_6}\) (highlighted in yellow) that covers the single-starred 1, and another essential prime implicant \(X_1 X_2\) that covers the double-starred one. The function has a single don’t-care cell, which is assigned 0 for the solution above. Under this assignment, the complement of the function also has a single IDF.

Fig. 17. Proof that the set of two variables \(\{X_5, X_8\}\) is a minimal set of supporting variables through discovery that \(f\) can be made independent of each of \(X_1, X_2, X_3, X_4, X_6\) and \(X_7\) but then it becomes minimally supported by members of that set.
Fig. 18. The Karnaugh map of Fig. 17 reduced to one of the two variables \( (X_5, X_8) \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with three of the 0’s combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant \( X_5 \overline{X_8} \) (highlighted in yellow) that covers the starred 1, and another essential prime implicant \( \overline{X_5} X_8 \) (highlighted in green) that covers the double-starred 1. The complement of the function also has a single IDF.

Fig. 19. Proof that the set of two variables \( \{X_6, X_7\} \) is a minimal set of supporting variables through discovery that \( f \) can be made independent of each of \( X_1, X_2, X_3, X_4, X_5 \) and \( X_8 \) but then it becomes minimally supported by members of that set.

Fig. 20. The Karnaugh map of Fig. 19 reduced to one of the two variables \( (X_6, X_7) \). This map inherits the three asserted high (1) configurations (with two of the 1’s combined), and the four asserted low (0) configurations of the original map (with two of the 0’s combined, and the other two 0’s combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant \( X_6 X_7 \) (highlighted in yellow) that covers the starred 1, and another essential prime implicant \( \overline{X_6} \overline{X_7} \) (highlighted in green) that covers the double-starred 1.
Fig. 21. An attempt to prove that the set of two variables \( \{X_1, X_2\} \) is a minimal set of supporting variables is doomed to failure. The figure shows that \( f \) can be made independent of each of \( X_3, X_4, X_6, X_7 \) and \( X_8 \) but then it becomes impossible to render it independent of \( X_5 \) because the two half maps across the \( X_5 \) borders (highlighted in red) cannot be matched.

Fig. 22. Attempt to prove that the set of four variables \( \{X_1, X_2, X_6, X_7\} \) is a minimal set of supporting variables. The function \( f \) can be made independent of each of \( X_3, X_4, X_7 \) and \( X_8 \) as shown but the reduced map of \( f \) in terms of \( X_1, X_2, X_6 \) and \( X_8 \) does not exhibit minimality of the set of supporting variables.
Fig. 23. Illustration that the set of four supporting variables \{X_3, X_2, X_5, X_6\} is not a minimal one, as it can drop either \(X_6\) or \(X_5\), thereby being reduced to either the set \{X_1, X_2, X_5\} (same as in Figure 10) or the set \{X_1, X_2, X_6\} (same as in Fig. 16). These two later sets are minimal as each of them cannot drop further any variable.

5. CONCLUSIONS

We used a regular and modular eight-variable Karnaugh map to explore a large problem of cause-effect relations that mimics problems of the newly-established discipline of Qualitative Comparative Analysis (QCA). This problem involves an eight-variable partially-defined Boolean function (PDBF), that is dominantly unspecified. Without using the integer-programming approach, we devised a simple map procedure to discover minimal sets of supporting variables in the outset before proceeding to seek IDF representations. According to our scheme of first identifying minimal sets of supporting variables, we avoided the task of drawing prime-implicant loops on the initial eight-variable map, and postponed this task till the map was dramatically reduced in size.

The procedure outlined herein has important consequences for Qualitative Comparative Analysis (QCA). These consequences are not expected to be welcomed by the QCA community, since they clearly indicate that the too-often strong results claimed by QCA adherents need to be checked and scrutinized. While the problem studied herein yielded twenty three IDF solutions distributed under umbrellas of eight minimal sets of supporting variables, mainstream QCA would simply use a program of Boolean minimization that produces a single minimal sum selected from the aforementioned set of twenty three IDFs. That minimal sum would have been a wonderful solution, had it been appropriately justified. In our opinion, more observations have to be made in order to narrow down the possibilities and decrease the number of candidate IDFs. Otherwise, our attempt to give preference to a certain IDF over the others should be context-specific rather than tool-specific.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES


Rushdi AMA, Badawi RS. Handling partially-defined Boolean functions in Qualitative Comparative Analysis by deriving the minimal sets of supporting variables before identifying the irredundant sums, Journal of King Abdulaziz University: Faculty of Computers and Information Technology Sciences. 2021;10(2):1-44.


Rushdi AM, Rushdi MA. Mathematics and examples of the modern syllogistic method of propositional logic. In Ram, M. (Editor), Mathematics Applied in Information


